

Lie theory cheat sheet

Lie group \mathcal{M}, \circ		size	dim	$\mathcal{X} \in \mathcal{M}$	Constraint	$\boldsymbol{\tau}^\wedge \in \mathfrak{m}$	$\boldsymbol{\tau} \in \mathbb{R}^m$
Vector n -D	$\mathbb{R}^n, +$	n	n	$\mathbf{v} \in \mathbb{R}^n$	$\mathbf{v} - \mathbf{v} = \mathbf{0}$	$\mathbf{v} \in \mathbb{R}^n$	$\mathbf{v} \in \mathbb{R}^n$
Complex number	S^1, \cdot	2	1	$\mathbf{z} \in \mathbb{C}$	$\mathbf{z}^* \mathbf{z} = 1$	$i\theta \in i\mathbb{R}$	$\theta \in \mathbb{R}$
2D Rotation	$\text{SO}(2), \cdot$	4	1	\mathbf{R}	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$[\theta]_\times = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} \in \mathfrak{so}(2)$	$\theta \in \mathbb{R}$
2D Rigid Motion	$\text{SE}(2), \cdot$	9	3	$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix}$	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$\begin{bmatrix} [\theta]_\times & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(2)$	$\begin{bmatrix} \boldsymbol{\rho} \\ \theta \end{bmatrix} \in \mathbb{R}^3$
Quaternion	S^3, \cdot	4	3	$\mathbf{q} \in \mathbb{H}$	$\mathbf{q}^* \mathbf{q} = 1$	$\boldsymbol{\theta}/2 \in \mathbb{H}_p$	$\boldsymbol{\theta} \in \mathbb{R}^3$
3D Rotation	$\text{SO}(3), \cdot$	9	3	\mathbf{R}	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$[\boldsymbol{\theta}]_\times = \begin{bmatrix} 0 & -\theta_z & \theta_y \\ \theta_z & 0 & -\theta_x \\ -\theta_y & \theta_x & 0 \end{bmatrix} \in \mathfrak{so}(3)$	$\boldsymbol{\theta} \in \mathbb{R}^3$
3D Rigid Motion	$\text{SE}(3), \cdot$	16	6	$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix}$	$\mathbf{R}^\top \mathbf{R} = \mathbf{I}$	$\begin{bmatrix} [\boldsymbol{\theta}]_\times & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3)$	$\begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix} \in \mathbb{R}^6$

Operation	Inverse	Compose	Exp	Log	Right- \oplus	Right- \ominus
Right Jacobians	$\mathbf{J}_\chi^{-1} = -\mathbf{Ad}_\chi$	$\begin{matrix} \mathbf{J}_\chi^{\chi \circ \gamma} = \mathbf{Ad}_{\gamma^{-1}} \\ \mathbf{J}_\gamma^{\chi \circ \gamma} = \mathbf{I} \end{matrix}$	$\mathbf{J}_\tau^{\text{Exp}(\tau)} = \mathbf{J}_r(\tau)$	$\mathbf{J}_\chi^{\text{Log}(\chi)} = \mathbf{J}_r^{-1}(\tau)$	$\begin{matrix} \mathbf{J}_\chi^{\chi \oplus \tau} = (\mathbf{Ad}_{\text{Exp}(\tau)})^{-1} \\ \mathbf{J}_\tau^{\chi \oplus \tau} = \mathbf{J}_r(\tau) \end{matrix}$	$\begin{matrix} \mathbf{J}_\chi^{\gamma \ominus \chi} = -\mathbf{J}_l^{-1}(\tau) \\ \mathbf{J}_\tau^{\gamma \ominus \chi} = \mathbf{J}_r^{-1}(\tau) \end{matrix}$

Note: In accordance to `manif` implementation, all Jacobians in this document are **right Jacobians**, whose definition reads: $\frac{\delta f(X)}{\delta X} = \lim_{\varphi \rightarrow 0} \frac{f(X \oplus \varphi) \ominus f(X)}{\varphi}$.

However, notice that one can relate the left- and right- Jacobians with the Adjoint, $\frac{\varepsilon \partial f(\chi)}{\partial \chi} \mathbf{Ad}_\chi = \mathbf{Ad}_{f(\chi)} \frac{\chi \partial f(\chi)}{\partial \chi}$, see [1] Eq. (46).

[1] J. Solà, J. Deray, and D. Atchuthan, “A micro Lie theory for state estimation in robotics,” Tech. Rep. IRI-TR-18-01, Institut de Robòtica i Informàtica Industrial, Barcelona, 2018. Available at arxiv.org/abs/1812.01537.

$\mathcal{M}, \circ \backslash \text{Op}$	Identity	Inverse	Compose	Act	Exp	Log
$\mathbb{R}^n, +$	$\mathbf{v} = [\mathbf{0}]$	$-\mathbf{v}$	$\mathbf{v}_1 + \mathbf{v}_2$	$\mathbf{v} + \mathbf{p}$	\mathbf{v}	\mathbf{v}
S^1, \cdot	$z = 1 + i0$	z^*	$z_1 \ z_2$	$z \ v$	$z = \cos \theta + i \sin \theta$	$\theta = \arctan2(\text{Im}(z), \text{Re}(z))$
$\text{SO}(2), \cdot$	$\mathbf{R} = \mathbf{I}$	$\mathbf{R}^{-1} = \mathbf{R}^\top$	$\mathbf{R}_1 \ \mathbf{R}_2$	$\mathbf{R} \cdot \mathbf{v}$	$\mathbf{R} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$	$\theta = \arctan2(r_{21}, r_{11})$
$\text{SE}(2), \cdot$	$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} = \mathbf{I}$	$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{R}^\top & -\mathbf{R}^\top \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$	$\mathbf{M}_1 \mathbf{M}_2 = \begin{bmatrix} \mathbf{R}_1 \mathbf{R}_2 & \mathbf{t}_1 + \mathbf{R}_1 \mathbf{t}_2 \\ \mathbf{0} & 1 \end{bmatrix}$	$\mathbf{M} \cdot \mathbf{p} = \mathbf{t} + \mathbf{R} \mathbf{p}$	$\mathbf{M} = \begin{bmatrix} \text{Exp}(\theta) & \mathbf{V}(\theta) \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix}^{(1)}$	$\boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{\rho} \\ \theta \end{bmatrix} = \begin{bmatrix} \mathbf{V}^{-1}(\theta) \mathbf{p} \\ \text{Log}(\mathbf{R}) \end{bmatrix}^{(1)}$
S^3, \cdot	$\mathbf{q} = 1 + i0 + j0 + k0$	$\mathbf{q}^* = w - ix - jy - jz$	$\mathbf{q}_1 \ \mathbf{q}_2$	$\mathbf{q} \mathbf{v} \mathbf{q}^*$	$\mathbf{q} = \cos \frac{\theta}{2} + \mathbf{u} \sin \frac{\theta}{2}$	$\boldsymbol{\theta} = 2\mathbf{v} \frac{\arctan2(\ \mathbf{v}\ , w)}{\ \mathbf{v}\ }$
$\text{SO}(3), \cdot$	$\mathbf{R} = \mathbf{I}$	$\mathbf{R}^{-1} = \mathbf{R}^\top$	$\mathbf{R}_1 \ \mathbf{R}_2$	$\mathbf{R} \cdot \mathbf{v}$	$\mathbf{R} = \mathbf{I} + \sin \theta [\mathbf{u}]_\times + (1 - \cos \theta) [\mathbf{u}]_\times^2$	$\boldsymbol{\theta} = \frac{\theta(\mathbf{R} - \mathbf{R}^\top)^\wedge}{2 \sin \theta}$
$\text{SE}(3), \cdot$	$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} = \mathbf{I}$	$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{R}^\top & -\mathbf{R}^\top \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$	$\mathbf{M}_1 \mathbf{M}_2 = \begin{bmatrix} \mathbf{R}_1 \mathbf{R}_2 & \mathbf{t}_1 + \mathbf{R}_1 \mathbf{t}_2 \\ \mathbf{0} & 1 \end{bmatrix}$	$\mathbf{M} \cdot \mathbf{p} = \mathbf{t} + \mathbf{R} \mathbf{p}$	$\mathbf{M} = \begin{bmatrix} \text{Exp}(\theta) & \mathbf{V}(\theta) \boldsymbol{\rho} \\ \mathbf{0} & 1 \end{bmatrix}^{(2)}$	$\boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{\rho} \\ \theta \end{bmatrix} = \begin{bmatrix} \mathbf{V}^{-1}(\theta) \mathbf{p} \\ \text{Log}(\mathbf{R}) \end{bmatrix}^{(2)}$

$\mathcal{M}, \circ \backslash \text{Ad/Jac}$	Ad	\mathbf{J}_r	\mathbf{J}_l	$\mathbf{J}_\mathbf{x}^{\mathbf{x} \cdot \mathbf{p}} \mid \mathbf{J}_\mathbf{p}^{\mathbf{x} \cdot \mathbf{p}}_{(\text{Act})}$
$\mathbb{R}^n, +$	$\mathbf{I} \in \mathbb{R}^{n \times n}$	\mathbf{I}	\mathbf{I}	$\mathbf{I} \quad \mathbf{I}$
S^1, \cdot	1	1	1	$\mathbf{R}[1]_\times \mathbf{v} \quad \mathbf{R}$
$\text{SO}(2), \cdot$	1	1	1	$\mathbf{R}[1]_\times \mathbf{v} \quad \mathbf{R}$
$\text{SE}(2), \cdot$	$\begin{bmatrix} \mathbf{R} & -[1]_\times \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$	$\begin{bmatrix} \sin \theta / \theta & (1 - \cos \theta) / \theta & (\theta \rho_1 - \rho_2 + \rho_2 \cos \theta - \rho_1 \sin \theta) / \theta^2 \\ (\cos \theta - 1) / \theta & \sin \theta / \theta & (\rho_1 + \theta \rho_2 - \rho_1 \cos \theta - \rho_2 \sin \theta) / \theta^2 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \sin \theta / \theta & (\cos \theta - 1) / \theta & (\theta \rho_1 + \rho_2 - \rho_2 \cos \theta - \rho_1 \sin \theta) / \theta^2 \\ (1 - \cos \theta) / \theta & \sin \theta / \theta & (-\rho_1 + \theta \rho_2 + \rho_1 \cos \theta - \rho_2 \sin \theta) / \theta^2 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \mathbf{R} & \mathbf{R}[1]_\times \mathbf{p} \end{bmatrix} \quad \mathbf{R}$
S^3, \cdot	$\mathbf{R}(\mathbf{q})$	$\mathbf{I} - \frac{1 - \cos \theta}{\theta^2} [\boldsymbol{\theta}]_\times + \frac{\theta - \sin \theta}{\theta^3} [\boldsymbol{\theta}]_\times^2$	$\mathbf{I} + \frac{1 - \cos \theta}{\theta^2} [\boldsymbol{\theta}]_\times + \frac{\theta - \sin \theta}{\theta^3} [\boldsymbol{\theta}]_\times^2$	$-\mathbf{R}(\mathbf{q}) [\mathbf{v}]_\times^{(3)} \quad \mathbf{R}(\mathbf{q})^{(3)}$
$\text{SO}(3), \cdot$	\mathbf{R}	$\mathbf{I} - \frac{1 - \cos \theta}{\theta^2} [\boldsymbol{\theta}]_\times + \frac{\theta - \sin \theta}{\theta^3} [\boldsymbol{\theta}]_\times^2$	$\mathbf{I} + \frac{1 - \cos \theta}{\theta^2} [\boldsymbol{\theta}]_\times + \frac{\theta - \sin \theta}{\theta^3} [\boldsymbol{\theta}]_\times^2$	$-\mathbf{R} [\mathbf{v}]_\times \quad \mathbf{R}$
$\text{SE}(3), \cdot$	$\begin{bmatrix} \mathbf{R} & [\mathbf{t}]_\times \mathbf{R} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}$	$\begin{bmatrix} \mathbf{J}_r(\boldsymbol{\theta}) & \mathbf{Q}(-\boldsymbol{\rho}, -\boldsymbol{\theta}) \\ \mathbf{0} & \mathbf{J}_r(\boldsymbol{\theta}) \end{bmatrix}^{(4)}$	$\begin{bmatrix} \mathbf{J}_l(\boldsymbol{\theta}) & \mathbf{Q}(\boldsymbol{\rho}, \boldsymbol{\theta}) \\ \mathbf{0} & \mathbf{J}_l(\boldsymbol{\theta}) \end{bmatrix}^{(4)}$	$\begin{bmatrix} \mathbf{R} & -\mathbf{R} [\mathbf{p}]_\times \end{bmatrix} \quad \mathbf{R}$

$$^{(1)} \mathbf{V}(\theta) = \frac{\sin \theta}{\theta} \mathbf{I} + \frac{1 - \cos \theta}{\theta} [1]_\times$$

$$^{(2)} \mathbf{V}(\theta) = \mathbf{I} + \frac{1 - \cos \theta}{\theta} [\mathbf{u}]_\times + \frac{\theta - \sin \theta}{\theta} [\mathbf{u}]_\times^2$$

$$^{(3)} \mathbf{R}(\mathbf{q}) = \begin{bmatrix} w^2 + x^2 - y^2 - z^2 & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & w^2 - x^2 + y^2 - z^2 & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & w^2 - x^2 - y^2 + z^2 \end{bmatrix}$$

$$^{(4)} \mathbf{Q}(\boldsymbol{\rho}, \boldsymbol{\theta}) = 1/2 [\boldsymbol{\rho}]_\times + \frac{\theta - \sin \theta}{\theta^3} ([\boldsymbol{\theta}]_\times [\boldsymbol{\rho}]_\times + [\boldsymbol{\rho}]_\times [\boldsymbol{\theta}]_\times + [\boldsymbol{\theta}]_\times [\boldsymbol{\rho}]_\times [\boldsymbol{\theta}]_\times) - \frac{1 - \frac{\theta^2}{2} - \cos \theta}{\theta^4} ([\boldsymbol{\theta}]_\times^2 [\boldsymbol{\rho}]_\times + [\boldsymbol{\rho}]_\times [\boldsymbol{\theta}]_\times^2 - 3 [\boldsymbol{\theta}]_\times [\boldsymbol{\rho}]_\times [\boldsymbol{\theta}]_\times) - \frac{1}{2} \left(\frac{1 - \frac{\theta^2}{2} - \cos \theta}{\theta^4} - 3 \frac{\theta - \sin \theta - \frac{\theta^3}{6}}{\theta^5} \right) ([\boldsymbol{\theta}]_\times [\boldsymbol{\rho}]_\times [\boldsymbol{\theta}]_\times^2 + [\boldsymbol{\theta}]_\times^2 [\boldsymbol{\rho}]_\times [\boldsymbol{\theta}]_\times)$$